X_i = ith eigenvector
dX_i = perturbation of the ith eigenvector
F_i = $K - \lambda_i M$
A_i = $[-dK + d\lambda_i M + \lambda_i dM]\{X_i\}$
n = size of F_i matrix
$d\bar{X}_i$ = dX_i with element "s" removed
$\bar{F}_i$ = F_i with column "s" and row "t" removed
$\bar{A}_i$ = A_i with element "t" removed
$\bar{M}_i$ = $X_{iT} dM X_i$

COLLINS and Thomson[1] calculate the variation of the eigenvalues of $d\lambda_i$ and the eigenvectors dX_i for the problem $KX_i = \lambda_i M_i X_i$. Although the given equation for the variation of the eigenvector dX_i is correct, it requires the prior calculation of all the other eigenvectors of the system. This is not difficult for small systems, but certainly is a problem when larger problems are considered. Also, since each eigenvector is linearly independent of all other eigenvectors, one would expect its perturbations to also be independent. This will now be shown to be true.

Fox and Kapoor[2] have, in fact, recently published equations to calculate the perturbations of the eigenvector without using the other eigenvectors. They report numerical difficulties, however. Simplification of their method, which will reduce the number of computations, should reduce roundoff error, and eliminate the near-singular condition which exists in their system, is possible.

The basic eigenvalue equation is

$$[K - \lambda_i M]\{X_i\} = 0 \tag{1}$$

Differentiating with each term considered as a variable produces

$$[K - \lambda_i M]\{dX_i\} = [-dK + d\lambda_i M + \lambda_i dM]\{X_i\} \tag{2}$$

The value $d\lambda_i$ may be calculated as Collins and Thomson state, and Eq. (2), for convenience, may be reduced to

$$[F_i]\{dX_i\} = \{A_i\} \tag{3}$$

If F_i is an $n \times n$ matrix and λ_i is a single root, then the rank of F_i is known to be $n - 1$. The vector dX_i then has no unique solution, but if a value is assumed for one element of dX_i, the others can be uniquely determined. If the original set of equations represented by matrix Eq. (3) is completely coupled, as is common for structures problems, each equation is consistent with the other equations. Therefore, any equation can be discarded. If some of the original equations are completely uncoupled, for example,

$$\begin{bmatrix} F_{1i} & 0 \\ 0 & F_{2i} \end{bmatrix} \begin{Bmatrix} dX_{1i} \\ dX_{2i} \end{Bmatrix} = \begin{Bmatrix} A_{1i} \\ A_{2i} \end{Bmatrix} \tag{3a}$$

then X_i will have the form

$$\begin{Bmatrix} X_{1i} \\ 0 \end{Bmatrix}$$

for a root of F_1 and

$$\begin{Bmatrix} 0 \\ X_{2i} \end{Bmatrix}$$

for a root of F_2. For a root of F_1, X_{2i} is always zero and hence dX_{2i} will always be zero by definition. Hence the element to be arbitrarily set zero must be a part of dX_{1i} and the equation to be discarded also selected from F_{1i}. The process is analogous to the fixing of a free-free stiffness matrix performed by structures analysts.

The obvious approach is to set one element of dX_i to zero, equivalent to normalizing the perturbed eigenvector equal to the unperturbed eigenvector at that element. The result is a consistent set of n equations in $n - 1$ unknowns.

In matrix notation the solution is

$$\{d_i\bar{X}\} = [\bar{F}_i]^{-1}\{\bar{A}_i\} \tag{4}$$

If it is desired to keep a constant generalized mass, Eq. (15) of Ref. 2 may be used. In our nomenclature it is

$$\begin{bmatrix} F_i \\ \hline 2X_{iT}[M] \end{bmatrix} \{dX_i\} = \begin{Bmatrix} A_i \\ \hline \bar{M}_i \end{Bmatrix} \tag{5}$$

Elimination of any row of F_i and the corresponding element of A_i, consistent with the coupling criteria discussed earlier, will yield a solvable set of equations. The process given involves fewer calculations and should give less roundoff than the more general solution given by Fox and Kapoor.

An alternative method of keeping a constant generalized mass is to first perform the solution of Eq. (4) then renormalize. Denoting the renormalized variation of the eigenvector by dX_i', the solution is

$$\{dX_i'\} = \left(\frac{\{X_i\}_T[M]\{X_i\}}{\{X_i + dX_i\}_T[M]\{X_i + dX_i\}}\right)^{1/2} \times \{X_i + dX_i\} - \{X_i\} \tag{6}$$

The advantage of solving Eqs. (4) and (6) over solving Eq. (5) is that elimination of the same row and column from the F_i matrix produces a symmetric matrix to be inverted. This should permit considerable decrease in computer time.

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Comment on "Angle-of-Attack Convergence and Windward-Meridian Rotation Rate of Rolling Re-Entry Vehicles"

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Nomenclature

$C_{N\alpha}$ = aerodynamic normal force derivative
C_{m_q} = aerodynamic pitch damping derivative
d = aerodynamic reference diameter
I = pitch or yaw moment of inertia
I_x = roll moment of inertia
l = static margin (distance of center of pressure aft of center of mass)
m = vehicle mass
M_η = aerodynamic pitch moment
M_ζ = aerodynamic yaw moment
p = roll rate
q = dynamic pressure
S = aerodynamic reference area
u = vehicle velocity
$\hat{v}$ = velocity component along Y axis

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$\hat{w}$ = velocity component along Z axis
α_t = total angle of attack
ζ = yaw axis
η = pitch axis
θ = total pitch angle (Euler angle)
μ = inertia ratio, I_x/I
ξ = roll axis
ϕ = roll orientation relative to plane of θ (Euler angle)
ψ = precession angle (Euler angle)

Subscripts

ζ = yaw
η = pitch
ξ = roll
$(\dot{\ })$ = time derivative

Introduction

IN Ref. 1, Platus describes the angular motion of a rolling re-entry vehicle in terms of the Euler angles, ψ, ϕ, θ, which determine the position of a set of body-fixed axes x, y, z, relative to nonrotating axes XYZ as shown in Fig. 1. The axes ξ, η, ζ are axes of roll, pitch and yaw, respectively, relative to the plane of the angle θ. Platus then assumes that the transverse aerodynamic moments consist only of pitch or yaw moments from angle of attack and pitch and yaw damping moments and gives the following moment equations:

$$M_\eta = -C_{N\alpha}qSl\theta - (qSd^2/2u)[-C_{m_q} + 2C_{N\alpha}I/md^2]\dot{\theta} \quad (1)$$

$$M_\zeta = -(qSd^2/2u)[-C_{m_q} + 2C_{N\alpha}I/md^2]\dot{\psi}\sin\theta \quad (2)$$

The normal force terms are inserted to account for lateral motion of the vehicle center of mass and are not justified any further in the paper.

The influence of normal force on the linearized angular motion of rolling missile has been considered by many authors,[2-4] but their results when placed in the form of Eqs. (1) and (2) differ significantly for the yaw moment equation

$$M_\zeta = -(qSd^2/2u)\{[-C_{m_q} + 2C_{N\alpha}I/md^2]\dot{\psi}\theta - 2C_{N\alpha}I_xp\theta/md^2\} \quad (3)$$

A comparison of Eqs. (2) and (3) show that Eq. (1) can only be valid for small roll rates for which the third term in Eq. (3) can be omitted. Exact analyses[4,5] have been performed for large values of θ and these indicate that even for zero roll rate Eqs. (1) and (2) are probably limited to small values of θ.

The influence of normal force on the angular motion of rolling re-entry vehicle can be quite important. It is, therefore, the purpose of this Comment to give a simple derivation of Eqs. (1) and (3).

Analysis

The total angle of attack lies in the plane containing the velocity vector and the missile's axis and is the angle between the velocity vector and the missile's axis. For small angles this can be easily calculated by the following equation:

$$\alpha_t \exp[i(\psi_t - \pi/2)] = \theta \exp[i(\psi - \pi/2)] - (\hat{v} + i\hat{w})/u \quad (4)$$

(The $-\pi/2$ appears in the exponentials due to the fact that ψ_t and ψ are angles between the Y-axis and the normals to the planes of α_t and θ respectively.) The linear pitch and yaw moments in the missile fixed coordinates can now be written in terms of the total angle of attack, its orientation and the angular velocity of the missile's axis,

$$M_\eta + iM_\zeta = -qSlC_{N\alpha}\{\alpha_t \exp[i(\psi_t - \psi)]\} + (qSd^2/2u)C_{m_q}(\dot{\theta} + i\dot{\psi}\theta) \quad (5)$$

In order to obtain Eq. (3) from Eq. (5) we must get a good linear approximation for α_t and ψ_t from Eq. (4).

If the only force acting is the linear aerodynamic normal force, the differential equation for the transverse velocity components can be easily obtained

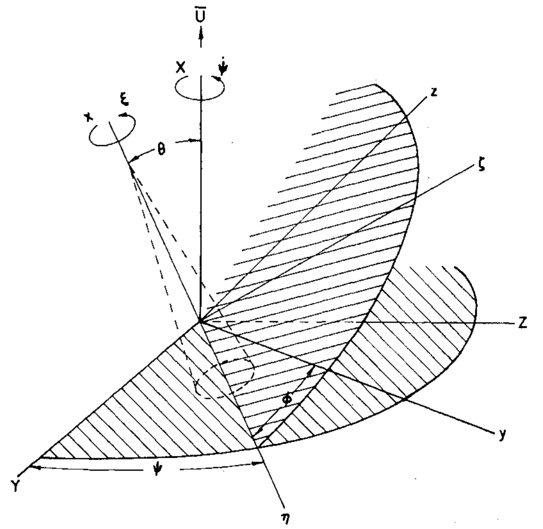

Fig. 1 Euler angles for three-degree-of-freedom rotational motion.

$$m[\dot{\hat{v}} + i\dot{\hat{w}}] = qSC_{N\alpha}\{\alpha_t \exp[i(\psi_t - \pi/2)]\} \quad (6)$$

A good first approximation to the pitching motion can be obtained by neglecting the damping moment and the normal force damping. For this case the linear pitching motion is a sum of the two possible coning motions with constant precessional rates $\dot{\psi}_+$ and $\dot{\psi}_-$

$$\theta \exp[i(\psi - \pi/2)] = K_+ \exp[i\psi_+] + K_- \exp[i\psi_-]$$

where

$$\begin{aligned}\dot{\psi}_+ + \dot{\psi}_- &= \mu p \\ \dot{\psi}_+ \cdot \dot{\psi}_- &= -C_{N\alpha}qSl/I\end{aligned} \quad (7)$$

The solution to Eq. (6) is the sum of a damped exponential and a steady-state response to $\theta e^{i\psi}$. We select the initial orientation of the X axis so that the damped exponential has an initial amplitude of zero.

Finally if the reasonable assumption $\dot{\psi}_{+,-} \gg qSC_{N\alpha}/mv$ can be made,

$$\begin{aligned}\hat{v} + i\hat{w} &= -iqSC_{N\alpha}[K_+ \exp(i\psi_+)/\dot{\psi}_+ + K_- \exp(i\psi_-)/\dot{\psi}_-]/m \\ &= (iI/ml)[\mu p(\theta e^{i(\psi - \pi/2)}) + i(d/dt)(\theta e^{i(\psi - \pi/2)})]\end{aligned} \quad (8)$$

The necessary relation for the total angle of attack follows from Eqs. (4) and (8)

$$\alpha_t \exp[i(\psi_t - \pi/2)] = \{\theta + (I/mlu) \times [\dot{\theta} + i\theta(\dot{\psi} - \mu p)]\}e^{i(\psi - \pi/2)} \quad (9)$$

Eqs. (5) and (9) now combine to yield Eq. (3), the yaw moment modified by the influence of the linear normal force on the lateral motion of the vehicle's center of mass.

Discussion

As can be seen from the analysis the third term in the yaw moment equation arises from the frequency shift induced by the gyroscopic spin, μp. This effect is safely neglected by Platus in his relation for angle of attack convergence of coning motion but is the cause of the "precession" instability which he considers in some detail. This instability is caused by the fact that the damping moment damps the slow frequency motion ("precessional" motion) less than it damps the fast frequency motion ("nutational" motion). The situation is very much changed when the proper damping effect of the normal force is considered. The relative damping of the two possible coning motions depends on the relative size of $C_{N\alpha}$ and C_{m_q} and either coning motion can display the instability discussed by Platus.

In Ref. 6 Phillips briefly considered the dynamic stability of spinning shell and makes the erroneous remark that all shells are dynamically unstable. In this case only the damping moment was considered. If the proper damping effect of the normal force had been inserted, he would have shown the correct result that most shells are dynamically stable. Thus damping induced by the normal force has a number of important consequences.

In both cases considered, a complete linear analysis should include drag and a linear Magnus moment coefficient. The Magnus coefficient combines with the additional normal force term in the yaw moment and can induce dynamic instability. Nonlinear Magnus moments can cause even more interesting behavior such as limit cycles and are discussed in detail in the literature.

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Reply by Author to C. H. Murphy

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Murphy points out that a small additional term should be included to describe completely normal force damping of the yaw motion. He then concludes that omission of this term is the cause of the precession instability discussed in Ref. 1. This instability is caused by the fact that negative precession motion is damped less than positive precession motion, as Murphy states above, but a second condition is also required for the instability to occur. The oscillation amplitude of the precession rate $\dot\psi$ must increase and/or the reduced roll rate p_r must increase such that the minimum value of the oscillation $y = \dot\psi - p_r$ approaches zero. The relative damping of the two precession motions depends on the roll acceleration $\dot p$ in addition to the relative size of $C_{N\alpha}$ and C_{m_q} (and Magnus effects, if present). Positive roll acceleration contributes to increased damping of positive precession and decreased damping of negative precession and, at the same time, causes the roll rate parameter p_r to increase and reach the lower envelope of the $\dot\psi$ oscillation, thereby inducing the instability discussed in Ref. 1. This instability can still occur if the complete effects of normal force damping are included, as shown below. It can also occur in the opposite direction, as Murphy points out. However, the importance of the damping induced by roll acceleration should be recognized. The quantitative relation between the various damping effects on the precession motion, including the complete effects of normal force damping and a linear Magnus moment coefficient, is derived here as an extension to Ref. 1.

The yaw moment equation, including the complete effects of normal force damping and a linear Magnus moment coefficient $C_{n_{p\alpha}}$, can be written

$$M_\zeta = -(qSd^2/2u)[(-C_{m_q} + C_{N_\alpha}')\dot\psi\theta - (\mu C_{N_\alpha}' + C_{n_{p\alpha}})p\theta] \tag{1}$$

where $\mu \equiv I_x/I$ and $C_{N_\alpha}' \equiv 2C_{N\alpha}I/md^2$. The small angle-of-attack pitch and yaw equations of motion can then be written

$$\begin{gathered}\ddot\theta + (\Omega^2 - y^2)\theta + \nu\dot\theta = 0\\ (\dot y + \nu y + \xi)\theta + 2y\dot\theta = 0\end{gathered} \tag{2}$$

where

$$\begin{gathered}\Omega \equiv (\omega^2 + p_r^2)^{1/2},\ y \equiv \dot\psi - p_r,\ \xi \equiv \dot p_r + \nu p_r - \nu_m p\\ \nu \equiv (qSd^2/2Iu)(-C_{m_q} + C_{N_\alpha}')\\ \nu_m \equiv (qSd^2/2Iu)(\mu C_{N_\alpha}' + C_{n_{p\alpha}})\end{gathered} \tag{3}$$

Equations (2) can be combined by eliminating θ to give the single equation in y and $v = \dot y$

$$dv/dy = [-4y^2(y^2 - \Omega^2) - (\nu^2y^2 - \xi^2) - 2y\xi + v(3v + 4\xi)]/(2yv) \tag{4}$$

suitable for phase plane analysis.

Stability of the two precession motions is determined by the character of the singularities at $v = 0$ and $y = \pm(\Omega - \epsilon)$, where $\epsilon \approx (\nu^2\Omega^2 \pm 2\xi\Omega - \xi^2)/(8\Omega^3)$ is small, in general, compared with Ω. Both singularities are of the spiral or center types with stability determined by the sign of the parameter ξ. For $\xi = 0$ both singularities are centers, indicating stable oscillations. For $\xi > 0$ the singularity at $y = \Omega - \epsilon$ is an unstable spiral and that at $y = -\Omega + \epsilon$ is a stable spiral, and for $\xi < 0$ the reverse is true. From the definition of ξ, Eq. (3), the stability criterion can be written

$$\begin{gathered}\xi = p_r[(\dot p/p) + \nu - 2(\nu_m/\mu)]\\ \xi = 0 \text{ both modes stable}\\ \xi > 0\begin{cases}\text{positive mode unstable}\\ \text{negative mode stable}\end{cases}\\ \xi < 0\begin{cases}\text{negative mode unstable}\\ \text{positive mode stable}\end{cases}\end{gathered} \tag{5}$$

This indicates that positive precessional motion grows and negative precessional motion decays for $\xi > 0$ and vice-versa for $\xi < 0$.† The only other singularities in the phase plane are an unstable node at $v = -\xi$ and an unstable saddle at $v = -\xi/3$ along the v axis. Consequently, an oscillation in $\dot\psi$ in either the positive or negative mode will persist until $y = \dot\psi - p_r$ reaches the limiting value $y = 0$, which is a necessary condition for the instability to occur. The precession instability so defined is a distinct reversal from the "positive precession mode" to the "negative precession mode," or vice-versa, and is not considered here to be merely a growth of one precession arm and a decay of the other. The fact that

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† A divergence of the $\dot\psi$ oscillation in the positive precession mode ($\xi > 0$) actually corresponds to a growth of the (negative) precession arm and a decay of the nutation arm in Murphy's notation; i.e., the transition is from the less stable to the more stable mode.